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The Isotropic Turbulent Mixer: Part II. Arbitrary Schmidt Number

STANLEY CORRSIN

The Johns Hopkins University, Baltimore, Maryland

One of the useful effects of turbulent fluid motion is its tendency to hasten the approach toward uniformity of a nonuniform fluid mixture. The stirring of cream into coffee is a familiar illustration. The simplest statistical measure of the departure from mixture uniformity at an instant is $\overline{c^2}(t)$, the mean-square fluctuation in contaminant concentration. To get a differential equation for $\overline{c^2}$, one begins with the mass transfer equation

$$\frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = D_v \nabla^2 C \qquad (1)$$

The diffusivity is taken constant, under restriction to low concentrations.

For a Newtonian fluid at very small Mach number with zero body forces, the corresponding momentum conservation statement is the Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} + u_K \frac{\partial u_i}{\partial x_K} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \qquad (2)$$

Total mass conservation with $\rho =$ constant is given by

$$\frac{\partial u_j}{\partial x_j} = 0 \qquad (3)$$

If one writes the concentration as the sum of mean and fluctuating parts, $C(\underline{\underline{x}}, t) = \overline{C} + c(\underline{\underline{x}}, t)$, then $\overline{c} \equiv 0$ by definition. Restricting to statistical homogeneity in space, $\overline{C} = \overline{C}(t)$ only, one sees that since there is no chemical reaction term in (1), $\overline{C} =$ constant. Therefore (1) reduces to an equation for c:

$$\frac{\partial c}{\partial t} + u_i \frac{\partial c}{\partial x_i} = D_v \nabla^2 c \qquad (4)$$

Multiplying (4) by c, averaging, using mass conservation to get u_i inside the derivative, and invoking homogeneity, one converts (4) to (*1*)

$$\frac{d}{dt}\overline{c^2} = -2 D_v \overline{\left(\frac{\partial c}{\partial x_j}\right)\left(\frac{\partial c}{\partial x_j}\right)} \qquad (5)$$

The corresponding decay equation for a typical velocity component in isotropic turbulence was given by Taylor (*2*):

$$\frac{d}{dt}\overline{u_1^2} = -2 \nu \overline{\left(\frac{\partial u_1}{\partial x_K}\right)\left(\frac{\partial u_1}{\partial x_K}\right)} \qquad (6)$$

Although the turbulent velocity does not appear explicitly in (5), its mixing action is of course felt through the concentration gradients; the random convection tends to steepen the gradients, thereby tending to increase the mean-square derivatives. This in turn accelerates molecular transport.

It is convenient, following Taylor, to replace the derivative products by defining characteristic lengths, the microscales, such that (5) and (6) are written as

$$\frac{d\overline{c^2}}{dt} = -12 D_v \frac{\overline{c^2}}{l^2} \qquad (7)$$

$$\frac{d\overline{u_1^2}}{dt} = -20 \nu \frac{\overline{u_1^2}}{\lambda_f^2} \qquad (8)$$

In general, l and λ_f are unknown functions of time, so these are indeterminate equations.

Dividing (7) by (8), and introducing root-mean-square values, one gets the relative decay equation (*1*)

$$\frac{\left(\dfrac{dc'}{c'}\right)}{\left(\dfrac{du_1'}{u_1'}\right)} = \frac{3}{5}\frac{1}{N_{Sc}}\left(\frac{\lambda_f}{l}\right)^2 \qquad (9)$$

The Schmidt number becomes Prandtl number if c is temperature fluctuation.

Theoretical estimates (*1, 3, 4, 5*) of the relative behaviors of scalar and velocity fields have been roughly confirmed by experiment for the particular case of heat as contaminant in air (*4*) (Prandtl number about 0.7). Batchelor (*6*) has recently pointed out that these theories are appropriate for Schmidt numbers of about unity or less, but not for values appreciably greater. In particular, the Oboukhov and Corrsin scalar spectral theories propose a maximum characteristic wave number

$$k^* = \left(\frac{\epsilon}{D_v^3}\right)^{1/4} \qquad (10)$$

somewhat in analogy to the Kolmogorov characteristic turbulence wave number:

$$k_K = \left(\frac{\epsilon}{\nu^3}\right)^{1/4} \qquad (11)$$

For $N_{Sc} >> 1$, Batchelor concluded that the maximum characteristic wave number of the scalar spectrum should be

$$k_B = \left(\frac{\epsilon}{\nu D_v^2}\right)^{1/4} \qquad (12)$$

He also deduced a characteristic spectral shape at wave numbers much larger than k_K. This includes the viscous-convective range, $k_K << k << k_B$, and in fact the entire viscous range $k >> k_K$.

For Schmidt numbers much smaller than unity, Batchelor, Howells, and Townsend (*7*) deduced a spectral shape for the inertial-diffusive range $k^* << k << k_K$.

One purpose of this paper is to improve the small Schmidt number analysis (*8*) by including the inertial-diffusive range in the concentration spectrum.

Another purpose is to use Batchelor's result to extend the previous decay estimate (*1, 4*) to the large Schmidt number case. This earlier theory was used some years ago to estimate the performance and scaling of a vastly simplified turbulent mixer (*8*). That report may be considered to be the first paper in a sequence of which this is the second. The same mixer problem is analyzed here for $N_{Sc} \leq 1$ and $N_{Sc} >> 1$.

A third paper in this sequence will include the effect of a passive, first-order chemical reaction.

DECAYING TURBULENCE $N_{SC} \leq 1$

In freely decaying isotropic turbulence, the relative decay rate of concentration and velocity fluctuations is estimated determining the microscale ratio on the right side of Equation (9). If $G(k,t)$ and $E(k,t)$ are the three-dimensional spectra of these two fields, such that

$$\overline{c^2} = \int_0^\infty G dk \quad \text{and} \quad \frac{1}{2}\overline{u^2} = \int_0^\infty E dk \qquad (13)$$

then l^2 and λ_f^2 can be written as

$$l^2 = 6\frac{\int_0^\infty G dk}{\int_0^\infty k^2 G dk} \quad \text{and} \quad \lambda_f^2 = 10\frac{\int_0^\infty E dk}{\int_0^\infty k^2 E dk} \qquad (14)$$

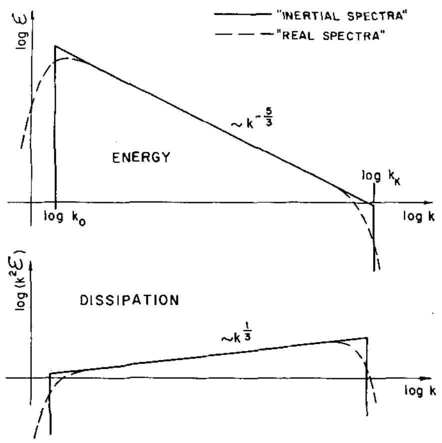

Fig. 1. Qualitative sketch of inertial spectra for turbulent motion.

Any reasonable theoretical or empirical forms for G and E will therefore permit the desired estimates. For large enough Reynolds numbers, it is found experimentally that an inertial subrange (*9, 10*) of Kolmogorov's theoretical type occurs in $E(k)$ (*11, 12, 13*):

$$E(k) = A\,\epsilon^{2/3}\,k^{-5/3} \qquad (15)$$

for

$$k_o << k << k_K$$

k_o, characterizing the energy-bearing part of the spectrum, is essentially the inverse of the integral scales [Equation (27)].

For still larger Reynolds numbers, this range should occupy a large fraction of the spectrum, and rough estimates of various properties can be made by replacing the entire spectrum by a purely inertial spectrum (*3, 14*), running all the way from k_o to k_K. Then for $k < k_o$ and $k > k_K$, $E = 0$. This notion is sketched qualitatively in Figure 1. It seems paradoxical to estimate the dissipation rate with an inertial shape of spectrum, but if the high k tail of the actual spectrum is exponential, as seems certain for decaying turbulence (*14*) and likely for stationary turbulence (*15*), then this predominantly viscous tail contributes relatively little to the second moment.

In some earlier applications of this method of estimation, it was applied to the one-dimentional spectra (*3, 4*). That should be less accurate because, ideally, they must have finite values at $k_1 = 0$ (*14*). For the one-dimensional spectrum which is the Fourier transform of the Kármán-Howarth $f(r)$ correlation function (that is for which λ_f is the Taylor microscale), it may be satisfactory to use a constant value in the range $0 \leq k_1 \leq k_o$. In fact, this form is consistent with the strictly inertial three-dimensional spectrum from k_o to k_K.

For large enough Peclet number $\left(\dfrac{u' l}{\sqrt{3} D_v} \text{ here}\right)$, one should be able to approximate $G(k)$ similarly. In the inertial-convection subrange one expects (5, 3)

$$G(k) = B\,\epsilon_c\,\epsilon^{-1/3}\,k^{-5/3} \qquad (16)$$

For $N_{Sc} \leq 1$, this theoretically lies in the range k_o, ${}_ck_o << k << k^*$. For $N_{Sc} << 1$, there is also an inertial-diffusive range (*7*) $k^* << k << k_K$, in which

$$G(k) \approx \frac{1}{3}\frac{\epsilon_c\,\epsilon^{2/3}}{D_v^3}\,k^{-17/3} \qquad (17)$$

A two-segment, power-law approximation to $G(k)$ follows the sketch of Figure 2, with (16) for ${}_ck_o \leq k \leq k^*$,

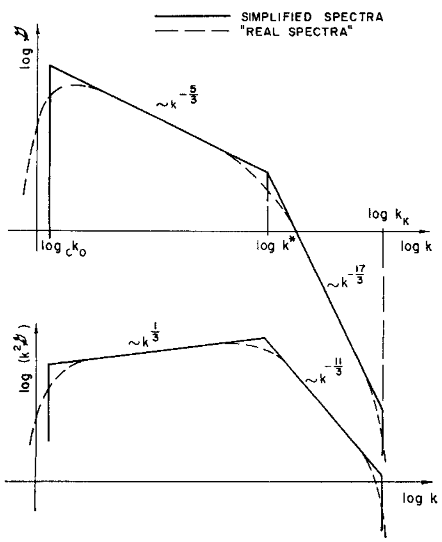

Fig. 2. Qualitative sketch of simplified scalar field spectra for $N_{SC} << 1$.

(17) for $k^* \leq k \leq k_K$, and $G = 0$ elsewhere.† In order to match levels at k^*

$$G(k) \approx B \frac{\epsilon_c \, \epsilon^{2/3}}{D_v^{\,3}} k^{-17/3} \tag{18}$$

with B instead of 1/3. B is presumably of order unity, so the order of magnitude is unchanged.

Substituting (15) into (14), one gets

$$\lambda_f^{\,2} \approx 20 \frac{k_o^{-2/3} - k_K^{-2/3}}{k_K^{4/3} - k_o^{4/3}} \tag{19}$$

Since $k_o << << k_K$, the smaller terms in the numerator and denominator are neglected, leaving

$$\lambda_f^{\,2} \approx \frac{20}{k_o^{2/3}\, k_K^{4/3}} \tag{20}$$

Substituting (16) and (18) into (14), one gets

$$l^2 \approx 12 \frac{({}_ck_o^{-2/3} - k^{*-2/3}) + \dfrac{1}{7}\dfrac{\epsilon}{D_v^{\,3}} (k^{*-14/3} - k_K^{-14/3})}{(k^{*4/3} - {}_ck_o^{4/3}) + \dfrac{1}{2}\dfrac{\epsilon}{D_v^{\,3}} (k^{*-8/3} - k_K^{-8/3})} \tag{21a}$$

This goes into clearer form if one expresses the second parts of the numerator and denominator in terms of k^* and N_{Sc} only:

$$l^2 \approx 12 \frac{({}_ck_o^{-2/3} - k^{*-2/3}) + \dfrac{1}{7} k^{*-2/3} (1 - N_{Sc}^{\,7/2})}{(k^{*4/3} - {}_ck_o^{4/3}) + \dfrac{1}{2} k^{*4/3} (1 - N_{Sc}^{\,2})} \tag{21b}$$

Since ${}_ck_o << << k^*$ and $N_{Sc} \leq 1$, this can be approximated by

$$l^2 \approx \frac{24}{(3 - N_{Sc}^{\,2})\; {}_ck_o^{2/3}\, k^{*4/3}} \tag{22}$$

With (20) and (22), using (10) and (11), one obtains

$$\left(\frac{\lambda_f}{l}\right)^2 \approx \frac{5}{6} (3 - N_{Sc}^{\,2})\, N_{Sc} \left(\frac{{}_ck_o}{k_o}\right)^{2/3} \tag{23}$$

applicable in the Schmidt number range $0 \leq N_{Sc} \leq 1$. Substituting this estimate into Equation (9) for the relative decay rate, and noting that $\dfrac{du_1'}{u_1'} = \dfrac{du'}{u'}$ for isotropic turbulence, one gets

$$\frac{\left(\dfrac{dc'}{c'}\right)}{\left(\dfrac{du'}{u'}\right)} \approx \frac{1}{2} (3 - N_{Sc}^{\,2}) \left(\frac{{}_ck_o}{k_o}\right)^{2/3} \tag{24}$$

This result reduces to the earlier form (*1, 3, 4*) in the case $N_{Sc} \approx 1$. Note its bounded dependence on N_{Sc} and its independence of Reynolds and Peclet numbers. The value of $\left(\dfrac{{}_ck_o}{k_o}\right)$ depends on the experimental arrangement.

At this point it should also be recalled that the theories of spectral shape used here, (15), (16), and (17), all require quasistationary fields. Their application to freely decaying fields implies some restrictions. For (15) these are discussed in Batchelor's book (9). For (16) the arguments are similar. For (17) they are discussed by Batchelor, Howells, and Townsend (7).

The only experiments even vaguely related to estimate (24) are those of reference 4, on the decay rates of temperature and velocity fluctuations behind a warm grid in air. Unfortunately, the turbulence Reynolds number ${}_\lambda N_{Re} \equiv \dfrac{u_1' \lambda}{\nu} = \dfrac{1}{\sqrt{6}} \dfrac{u' \lambda_f}{\nu}$ is only about 25. To make a comparison, it is convenient to relate k_o and ${}_ck_o$ to the (measured) integral scales. The general isotropic turbulence relation (9)

$$L_f = \frac{3\pi}{4} \frac{\int_o^\infty k^{-1} E dk}{\int_o^\infty E dk} \tag{25}$$

and the corresponding scalar field form

$$L_c = \frac{\pi}{2} \frac{\int_o^\infty k^{-1} G dk}{\int_o^\infty G dk} \tag{26}$$

are needed. The idealized spectra give

$$k_o \approx \frac{3\pi}{10} L_f^{-1} \quad \text{and} \quad {}_ck_o \approx \frac{\pi}{5} L_c^{-1} \tag{27}$$

This transforms (24) to

$$\frac{\left(\dfrac{dc'}{c'}\right)}{\left(\dfrac{du'}{u'}\right)} \approx \frac{1}{2} (3 - N_{Sc}^{\,2}) \left[\frac{2}{3}\frac{L_f}{L_c}\right]^{2/3} \tag{28}$$

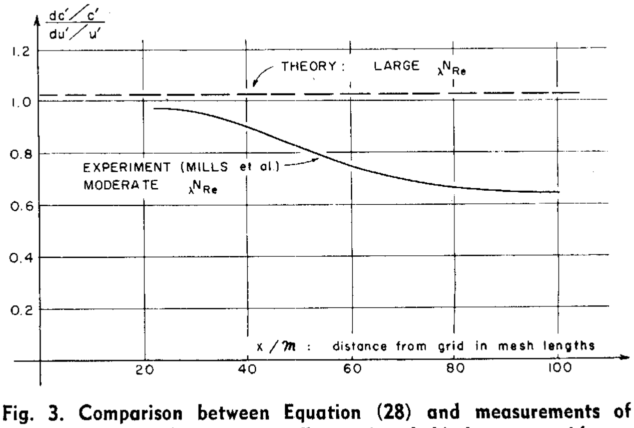

Fig. 3. Comparison between Equation (28) and measurements of turbulence and temperature fluctuations behind a warm grid.

† Supposing, for simplicity, that $k_o \lesssim {}_ck_o$.

With $N_{Sc} = 0.7$, the agreement with experiment [Figure 3] is fair, considering the mismatch of both Reynolds and Peclet numbers. It would be desirable to repeat these experiments at much larger Reynolds number and in mercury as well as air.

DECAYING TURBULENCE $N_{SC} >> 1$

For very large Schmidt numbers, the shape of G in the inertial-convective range is unchanged, but the scalar field cut off wave number is that given by Batchelor's form, Equation (12).

His theoretical prediction for the viscous range $k >> k_K$ is

$$G(k) \approx 2\,\epsilon_c \left(\frac{\nu}{\epsilon}\right)^{1/2} k^{-1} \exp\left\{-2\left(\frac{k}{k_B}\right)^2\right\} \quad (29)$$

If N_{Sc} is large enough, there will be a viscous-convective range $k_K << k << k_B$, in which the exponential is of order unity, so that

$$G(k) \approx 2\,\epsilon_c \left(\frac{\nu}{\epsilon}\right)^{1/2} k^{-1} \quad (30)$$

For the purpose of this paper, it will suffice to use this in estimating l^2. Figure 4 is a qualitative sketch of the actual and idealized spectra. In order to match at $k = k_K$, one uses

$$G(k) \approx B\,\epsilon_c \left(\frac{\nu}{\epsilon}\right)^{1/2} k^{-1} \quad (31)$$

instead of (30) for $k_K \leq k \leq k_B$.

Substituting (16) and (31) into (14), one finds

$$l^2 \approx 12 \frac{\epsilon^{-1/3}\left({}_ck_o^{-2/3} - k_K^{-2/3}\right) + \frac{2}{3}\left(\frac{\nu}{\epsilon}\right)^{1/2} \log\left(\frac{k_B}{k_K}\right)}{\epsilon^{-1/3}\left(k_K^{4/3} - {}_ck_o^{4/3}\right) + \frac{2}{3}\left(\frac{\nu}{\epsilon}\right)^{1/2}\left(k_B^2 - k_K^2\right)} \quad (32)$$

Neglecting $k_K^{-2/3}$ relative to ${}_ck_o^{-2/3}$, and ${}_ck_o^{4/3}$ relative to $k_K^{4/3}$, one has

$$l^2 \approx 12 \frac{{}_ck_o^{-2/3} k_K^{2/3} + \frac{2}{3}\log\left(\frac{k_B}{k_K}\right)}{k_K^2 + \frac{2}{3}\left(k_B^2 - k_K^2\right)} \quad (33)$$

Before simplifying further, one observes that as $N_{Sc} \to 1$, $k_B \to k_K$ and (33) reduces to (22), the estimate for $N_{Sc} \leq 1$.

For $N_{Sc} >>>> 1$, such that $k_B >>>> k_K$, (33) simplifies to

$$l^2 \approx 18 \frac{\left(\frac{k_K}{{}_ck_o}\right)^{2/3} + \frac{2}{3}\log\left(\frac{k_B}{k_K}\right)}{k_B^2} \quad (34)$$

but it is convenient here to work with the more detailed estimate in (33).

When k_B and k_K are expressed by (12) and (11), (33) can be rewritten as

$$l^2 \approx 36 \frac{{}_ck_o^{-2/3} k_K^{-4/3} + \frac{1}{3} k_K^{-2} \log N_{Sc}}{1 + 2 N_{Sc}} \quad (35)$$

With (20) and (35), the microscale ratio is

$$\left(\frac{\lambda_f}{l}\right)^2 \approx \frac{5}{9} \frac{1 + 2 N_{Sc}}{\left(\frac{k_o}{{}_ck_o}\right)^{2/3} + \frac{1}{3}\left(\frac{k_o}{k_K}\right)^{2/3} \log N_{Sc}} \quad (36)$$

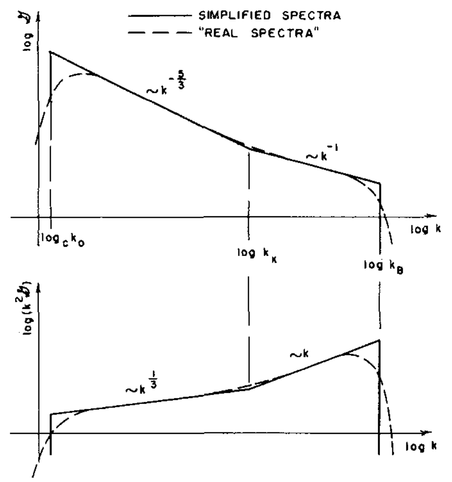

Fig. 4. Qualitative sketch of simplified scalar field spectra for $N_{SC} >> 1$.

and this gives the relative decay rate estimate [from (19)]:

$$\frac{\left(\frac{dc'}{c'}\right)}{\left(\frac{du'}{u'}\right)} \approx \frac{2 + N_{Sc}^{-1}}{3\left(\frac{k_o}{{}_ck_o}\right)^{2/3} + \left(\frac{k_o}{k_K}\right)^{2/3} \log N_{Sc}} \quad (37)$$

In contrast to the $N_{Sc} \leq 1$ case, one now has unbounded dependence on N_{Sc}, though weak. In further contrast, one finds dependence on Reynolds number [see (47) or (48)]. It is clear that the coefficient of the log N_{Sc} is small indeed.

$\left(\frac{k_o}{{}_ck_o}\right)$ can have a range of values, depending on the experimental configuration, but values of order 1 are perhaps the simplest to arrange.

A useful modified form can be gotten by estimating $\left(\frac{k_o}{k_K}\right)$ in terms of Reynolds number. A rough theoretical estimate for very large Reynolds numbers follows by substituting a purely inertial spectrum, Equation (15), into the expressions for dissipation rate

$$\epsilon = 2\nu \int_o^\infty k^2 E dk \approx 2A\nu\epsilon^{2/3} \int_{k_o}^{k_K} k^{1/3} dk \quad (38)$$

and turbulent kinetic energy per unit mass:

$$\frac{1}{2}\overline{u^2} = \int_o^\infty E dk \approx A\epsilon^{2/3} \int_{k_o}^{k_K} k^{-5/3} dk \quad (39)$$

From (38), neglecting $k_o^{4/3}$ relative to $k_K^{4/3}$, one gets the Kolmogorov constant $A \approx 2/3$ (14). Then, from (39), neglecting $k_K^{-2/3}$ relative to $k_o^{-2/3}$, one gets

$$\overline{u^2} \approx 2\epsilon^{2/3} k_o^{-2/3} \quad (40)$$

Using the definition $k_K \equiv \left(\frac{\epsilon}{\nu^3}\right)^{1/4}$ to eliminate ϵ one obtains

$$\frac{k_o}{k_K} \approx \frac{\left(\frac{2}{3}\right)^{3/8}}{{}_oN_{Re}^{3/4}} \quad (41)$$

in which

$${}_oN_{Re} \equiv \frac{u'}{\sqrt{3}\,\nu\,k_o} \quad (42)$$

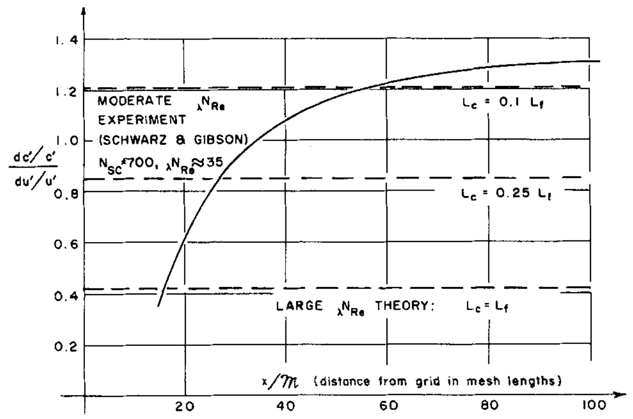

Fig. 5. Comparison between Equation (51) and measurements of turbulence and salinity fluctuations behind a grid.

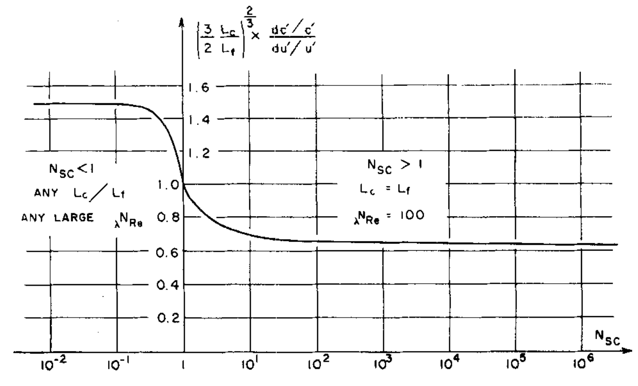

Fig. 6. Relative decay rate as a function of Schmidt number, illustrative example.

is essentially a turbulence Reynolds number based on an integral scale.

Since earlier work in this domain has used

$$_{\lambda}N_{Re} \equiv \frac{u_1'\lambda}{\nu} = \frac{u'\lambda_f}{\sqrt{6}\,\nu} \tag{43}$$

ϵ is alternately eliminated in (40) by using Taylor's form

$$\epsilon = 10\,\nu\,\frac{\overline{u^2}}{\lambda_f^2} \tag{44*}$$

to get

$$_{o}N_{Re} = \frac{\sqrt{3}}{5}\,{_{\lambda}N_{Re}}^2 \tag{45†}$$

so that

$$\frac{k_o}{k_K} \approx \frac{\left(\frac{20}{3}\right)^{3/4}}{_{\lambda}N_{Re}^{3/2}} \tag{46}$$

This expression puts (37) into the form

$$\frac{\left(\frac{dc'}{c'}\right)}{\left(\frac{du'}{u'}\right)} \approx \frac{2 + N_{Sc}^{-1}}{3\left(\frac{k_o}{_{c}k_o}\right)^{2/3} + \left(\frac{20}{3}\right)^{1/2} {_{\lambda}N_{Re}^{-1}} \log N_{Sc}} \tag{47}$$

This permits a reasonable estimate of the second denominator term relative to the first. Since it appears virtually negligible, consider a case for which it is relatively large; suppose $N_{Sc} = 10^5$ and $_{\lambda}N_{Re} = 10^3$ [near the lower limit for which one may expect the Kolmogorov inertial subrange theory to apply (*13*)]. Then $\left(\frac{20}{3}\right)^{1/2} {_{\lambda}N_{Re}^{-1}} \log N_{Sc} \doteq 3 \times 10^{-2}$, quite negligible in this approximation unless $k_o << {_{c}k_o}$. The second numerator term is also negligible.

It is concluded that for $N_{Sc} >> 1$, $_{\lambda}N_{Re} >> 1$, and $k_o <\!\!\setminus\!\!< {_{c}k_o}$

$$\frac{\left(\frac{dc'}{c'}\right)}{\left(\frac{du'}{u'}\right)} \approx \frac{2}{3}\left(\frac{_{c}k_o}{k_o}\right)^{2/3} \tag{48}$$

which is independent of N_{Sc}.

* The coefficient of this equation was copied wrong in *I* (*8*) (ϕ there is ϵ here; q^2 there is u^2 here). The coefficient of Equation (11) there should be 15, with corresponding changes in (12) and (13).

† In this connection, it should be noted that Equation (27) connecting k_o with the integral scale L_f implies $k_o \approx \frac{3\pi}{20} L_g^{-1}$, so that $\frac{\lambda}{L_g} \approx \frac{17}{_{\lambda}N_{Re}}$ as a more specific estimate than Equation (43) of reference *14*. It is in reasonable agreement with experiment.

It must be emphasized that since the left side of (47) is the ratio of logarithmic differentials, the right side is actually an exponent in the integrated equation. If the actual ratio $\frac{c'/c'(0)}{u'/u'(0)}$ is to be predicted for large times, small differences in exponent can be important.

For future comparison with experiment it may be convenient to replace k_o and $_{c}k_o$ in terms of the corresponding integral scales. The first part of (27) applies to the turbulent motion, but for large Schmidt number one must put the appropriate spectrum into (26) for the scalar field. With (16) and (31) in their appropriate wave number ranges, the formal calculation gives

$$L_c \approx \frac{\pi}{5}\,{_{c}k_o^{-1}} \left\{ \frac{1 - \left(\frac{_{c}k_o}{k_K}\right)^{5/3} + \frac{5}{3}\left(\frac{_{c}k_o}{k_K}\right)^{5/3}[1 - N_{Sc}^{-1/2}]}{1 - \left(\frac{_{c}k_o}{k_K}\right)^{2/3} + \frac{1}{3}\left(\frac{_{c}k_o}{k_K}\right)^{2/3} \log N_{Sc}} \right\} \tag{48}$$

after replacement of k_B/k_K in terms of N_{Sc}. Since $_{c}k_o <<<< k_K$

$$L_c \approx \frac{\frac{\pi}{5}\,{_{c}k_o^{-1}}}{1 + \frac{1}{3}\left(\frac{_{c}k_o}{k_K}\right)^{2/3} \log N_{Sc}} \tag{49}$$

The second term in the denominator is also negligible for an extremely wide range of Schmidt numbers. Suppose, for example, $N_{Sc} = 10^6$. If $_{c}k_o = 10^{-4}\,k_K$ (a choice corresponding to $_{c}k_o \approx k_o$ and $_{\lambda}N_{Re} \approx 10^3$, which is easily attainable in liquid mixers), this second term is only 0.4%. In cases for which this term is neglected, one can again use

$$L_c \approx \frac{\pi}{5}\,{_{c}k_o^{-1}} \tag{27}$$

Then (48) can be written as

$$\frac{\left(\frac{dc'}{c'}\right)}{\left(\frac{du'}{u'}\right)} \approx \left(\frac{2}{3}\right)^{5/3}\left(\frac{L_f}{L_c}\right)^{2/3} \tag{50}$$

On the other hand, if N_{Sc} is so large (at a modest $_{\lambda}N_{Re}$) that the log term must be kept in (47), then the transformation to integral scale, Equation (49), must keep its log term too. With some further approximations based on the restriction that the log terms are small, though not negligible, one can write the more detailed form as

$$\frac{\left(\frac{dc'}{c'}\right)}{\left(\frac{du'}{u'}\right)} \approx \frac{\left(\frac{2}{3}\right)^{5/3}\left(\frac{L_f}{L_c}\right)^{2/3}}{1 + 1.1\left(\frac{L_f}{L_c}\right)^{2/3} {}_\lambda N_{Re}^{-1} \log N_{Sc}} \qquad (51)$$

In this case the N_{Sc}^{-1} term of (47) is negligible. When the log term in (51) is appreciably smaller than unity, (51) is a close approximation to (47).

The only published data on decaying isotropic fields at large N_{Sc} are those of Gibson and Schwarz (*16*) at a Reynolds number (${}_\lambda N_{Re} \approx 35$) too low for good applicability of this theory. They released salt water from a regular grid spanning the test section of a water tunnel ($N_{Sc} \approx 700$). Since the saline sources were point sources (distributed along each rod at one per mesh length), while the momentum deficiency sources (the rods) were each two dimensional, there is no good way of guessing at $\left(\frac{L_f}{L_c}\right)$. Figure 5 is a comparison of experiment with Equation (51), computed with $\frac{L_f}{L_c} = 1$, 4, and 10. The $\frac{du'}{u'}$ is taken from data of Townsend behind a similar grid in air at the same Reynolds number (*9*).

Figure 6 is a plot of theoretical relative decay rate as a function of N_{Sc}, basically Equations (28) and (51), with a connecting region for $N_{Sc} \geqq 1$, where the full (33) is used instead of (34). Then in place of (51)

$$\frac{\left(\frac{dc'}{c'}\right)}{\left(\frac{du'}{u'}\right)} \approx \frac{2 + N_{Sc}^{-1}}{3\left(\frac{3}{2}\right)^{2/3}\left(\frac{L_c}{L_f}\right)^{2/3} + 2\left(\frac{5}{3}\right)^{3/2} {}_\lambda N_{Re}^{-1} \log N_{Sc}} \qquad (52)$$

Since (52) is only slightly more complicated than (51), why shouldn't (52) be used generally? The answer is that (52) is not based on a fully consistent approximation; Batchelor's theory (*6*) is designed for $N_{Sc} >> 1$, not the more general condition $N_{Sc} \geqq 1$. Since (51) matches (28) at $N_{Sc} = 1$, however, it may actually turn out to be useful, but such success will be a fortuitous consequence of the fact that $k^* = k_K = k_B$ for $N_{Sc} = 1$.

ISOTROPIC MIXER WITH $N_{SC} \leqq 1$

Following the ideas in I (*8*), one postulates the existence of a mixer containing stationary, isotropic turbulence. Stationarity and isotropy are, of course, incompatible restrictions (*9*). But at large enough Reynolds numbers in any turbulent motion one expects local isotropy in the sense of Kolmogorov (*9*). At $t = 0$ there is an isotropic c field, whose subsequent decay is sought. The principal idea of this theory is that with λ_f constant, l will also be constant, even though $\overline{c^2}$ is changing. In other words, it is assumed that stationary estimates like (23) and (36) apply.

In the two previous sections, the stationary estimates were used when both $\overline{u^2}$ and $\overline{c^2}$ were changing. In the present example, the λ_f estimate is actually better, but the l estimate may be worse because the maintained turbulence destroys $\overline{c^2}$ more rapidly.

With l constant, Equation (7) integrates as

$$\overline{c^2}(t) = \overline{c_o^2} \exp\left\{-\frac{12\,D_v\,t}{l^2}\right\} \qquad (53)$$

$\overline{c_o^2} \equiv \overline{c^2}(0)$. If one wishes to introduce the notion of a homogenization time constant, a typical choice would be

$$\tau \equiv \frac{l^2}{12\,D_v} \qquad (54)$$

that is

$$\overline{c^2}(t) = \overline{c_o^2} \exp\left\{-\frac{t}{\tau}\right\} \qquad (55)$$

For the case $N_{Sc} \leqq 1$, Equation (22) gives a convenient estimate of l^2. Replacing k^* by its definition [Equation (10)], one gets

$$\tau \approx \frac{2}{(3 - N_{Sc}^2)\,\epsilon^{1/3}\,{}_c k_o^{2/3}} \qquad (56)$$

With (27), ${}_c k_o$ can be replaced in terms of the integral scale:

$$\tau \approx \frac{2}{(3 - N_{Sc}^2)}\left(\frac{5}{\pi}\right)^{2/3}\frac{L_c^{2/3}}{\epsilon^{1/3}} \qquad (57)$$

Alternatively, ϵ can be eliminated from (56) by using (20) in the form

$$\lambda_f^2 \approx \frac{20}{k_o^{2/3}}\left(\frac{\nu}{\epsilon^{1/3}}\right) \qquad (20a)$$

The result is

$$\tau \approx \frac{\lambda_f^2}{10\nu\,(3 - N_{Sc}^2)}\left(\frac{k_o}{{}_c k_o}\right)^{2/3} \qquad (58)$$

a generalization of Equation (10) in I. For simplicity, ${}_c k_o$ was chosen equal to k_o in I, but this need not be so. ${}_c k_o$ is an inverse measure of the size and spacing of the blobs of c fed into the mixer. As may be expected intuitively, a faster approach to uniformity results with smaller blobs (larger ${}_c k_o$). The effect of changing the turbulence structure is not immediately clear because λ_f and k_o are both involved. This double effect can be sorted out into a size effect and a Reynolds number effect by eliminating ϵ between (40) and (44) to get

$$\lambda_f^2 \approx \frac{100\sqrt{2}}{3\,{}_\lambda N_{Re}^2\,k_o^2} \qquad (59)$$

This transforms (58) into

$$\tau \approx \frac{10\sqrt{2}}{3\nu\,(3 - N_{Sc}^2)\,{}_\lambda N_{Re}^2\,{}_c k_o^{2/3}\,k_o^{4/3}} \qquad (60)$$

For a prescribed contaminant and scalar size structure, the fluid mechanical effects are explicitly displayed by selecting $(\tau\,D_v\,{}_c k_o^2)$ as a dimensionless time, isolating the ${}_\lambda N_{Re}$ and scale ratio on the right side:

$$N_{Sc}\,(3 - N_{Sc}^2)\,\tau\,D_v\,{}_c k_o^2 \approx \frac{10\sqrt{2}}{3}\,{}_\lambda N_{Re}^{-2}\left(\frac{{}_c k_o}{k_o}\right)^{4/3} \qquad (61)$$

Next consider the requirements for the scaling of this simple type of mixer, especially for power requirements. For very large Reynolds numbers, the efficiency η of turbulence generation should be virtually independent of Reynolds number, so the power input at mixer shaft or jet orifice is

$$P = \eta\,M\,\epsilon \qquad (62)$$

Eliminating ϵ between (57) and (62) one gets the time constant almost entirely in nonturbulence parameters, a more useful engineering expression:

$$\tau \approx \frac{2\eta}{3 - N_{Sc}^2}\left(\frac{5}{\pi}\right)^{2/3}\frac{L_c^{2/3}\,M^{1/3}}{P^{1/3}} \qquad (63)$$

Writing the same estimate for a geometrically similar mixer (primed variables) one can (for example) ask for the power required to give equal mixing times. If $\eta' = \eta$, the power ratio such that $\tau' = \tau$ is

$$\frac{P'}{P} = \left(\frac{3 - N_{Sc}^2}{3 - N_{Sc}'^2}\right) K^3\,K_c^3 \qquad (64)$$

where K and K_c are scaling factors for flow scale and contaminant fluctuation scale, respectively:

$$M' \equiv K^3 M; \quad L_c' = K_c L_c \tag{65}$$

One restricts to $\rho' = \rho$, so that K scales the flow system:

$$H' = K H \tag{66}$$

Since one is restricted to large Reynolds and Peclet numbers, the integral scales of the fluctuation fields are roughly proportional to the gross dimensions of the mean (gradient) fields, hence the gross flow dimensions and the contaminant injection system dimensions, respectively ($L_f \sim H$, and $L_c \sim H_c$). Therefore K_c scales the latter

$$H_c' = K_c H_c \tag{67}$$

and (64) may be interpreted in terms of gross system geometry, without direct reference to turbulence structure.

In the special case of $N_{Sc}' = N_{Sc}$ and $K_c = K$, (64) reduces to Equation (20) of I. The second condition should be well fulfilled, for example, if the turbulence is generated primarily by jets of contaminant.

ISOTROPIC MIXER WITH $N_{SC} >> 1$

For very large Schmidt numbers, l^2 in (54) or (55) is estimated with (34) instead of (22). Then

$$\tau \approx \frac{1}{2} \left\{ \frac{3}{\epsilon^{1/3}\, {}_c k_c^{2/3}} + \left(\frac{\nu}{\epsilon}\right)^{1/2} \log N_{Sc} \right\} \tag{68}$$

Unlike the $N_{Sc} \leqq 1$ case, this has unbounded dependence on N_{Sc}. Nevertheless it is weak.

Equation (68) does not go to (56) as $N_{Sc} \to 1$ because it is an asymptotic approximation. If one were to work from (33) instead of (34) [hence (35)], the more general result would provide smooth interpolation:

$$\tau \approx \frac{1}{(2 + N_{Sc}^{-1})} \left\{ \frac{3}{\epsilon^{1/3}\, {}_c k_c^{2/3}} + \left(\frac{\nu}{\epsilon}\right)^{1/2} \log N_{Sc} \right\} \tag{69}$$

For an explanation of why (69) should be regarded as a convenient interpolation formula rather than a straightforward theoretical result, see the remarks following Equation (52).

In terms of the contaminant integral scale, (68) is

$$\tau \approx \frac{1}{2} \left\{ 4 \frac{L_c^{2/3}}{\epsilon^{1/3}} + \left(\frac{\nu}{\epsilon}\right)^{1/2} \log N_{Sc} \right\} \tag{70*}$$

Here $\left(\dfrac{5}{\pi}\right)^{2/3}$ has been replaced by $\dfrac{4}{3}$ for simplicity.

Evidently the scaling analysis for $N_{Sc} >> 1$ is more complex than for $N_{Sc} \leqq 1$. Write

$$\nu' = R\,\nu; \quad \epsilon' = Q\,\epsilon; \quad L_c' = K_c L_c \tag{71}$$

One can postpone consideration of total power and look for relations among R, Q, and K_c such that $\tau' = \tau$; that is

$$4\left(\frac{K_c^2}{Q}\right)^{1/3} \left(\frac{L_c^2}{\epsilon}\right)^{1/3} + \left(\frac{R}{Q}\right)^{1/2} \left(\frac{\nu}{\epsilon}\right)^{1/2} \log N_{Sc}' \approx 4\left(\frac{L_c^2}{\epsilon}\right)^{1/3} + \left(\frac{\nu}{\epsilon}\right)^{1/2} \log N_{Sc} \tag{72}$$

Since L_c is independent of ν and ϵ, the coefficients of the two kinds of terms must be equated separately:

$$Q = K_c^2 \text{ and } R = Q \left[\frac{\log N_{Sc}}{\log N_{Sc}'}\right]^2 \tag{73}$$

* J. Lee and R. S. Brodkey (*A.I.Ch.E. Journal*, March, 1964) indicate that this may agree roughly with experiment.

Power requirements are introduced via Equation (62) and the mixer size factor K via (65).

From (62)

$$\frac{P'}{P} = \frac{\eta' M' \epsilon'}{\eta M \epsilon} = \frac{\eta'}{\eta} K^3 Q \tag{74}$$

If the turbulence generating efficiencies are the same in both mixers, a plausible estimate if they are geometrically similar and are both at large Reynolds numbers is

$$\frac{P'}{P} = K^3 Q \tag{75}$$

Replacing Q in terms of K_c according to (73), one has the simultaneous conditions for $\tau' = \tau$:

$$\frac{P'}{P} = K^3 K_c^2 \text{ and } R = K_c^2 \left[\frac{\log N_{Sc}}{\log N_{Sc}'}\right]^2 \tag{76}$$

If $\rho' = \rho$, K scales the characteristic lengths of the flow field. Note that this scaling prescription gives a power scaling identical with the $N_{Sc}' = N_{Sc}$ limit of the $N_{Sc} \leqq 1$ case, Equation (64).

In the more restricted but technologically useful case in which fluid and contaminant are the same in the two mixers, $R = 1$, $N_{Sc}' = N_{Sc}$, and one need not equate separately the two kinds of terms in (72). This statement of $\tau' = \tau$ can then be written as

$$K_c Q^{-1/3} \approx [1 + K_c^{-2/3} Q^{-1/6} \beta]^{-1} (1 + \beta)^{-1} \tag{77}$$

where

$$\beta \equiv \frac{1}{4} \left(\frac{\nu^{1/2} \log N_{Sc}}{\epsilon^{1/6} L_c^{2/3}}\right)$$

This is a cubic equation for $Q^{-1/6}$.

A useful asymptotic case is that for which

$$\beta << 1 \tag{78}$$

Then the binomial expansions of the factors in (77), with neglect of terms squared and higher order in β, yields a quadratic equation for $Q^{-1/6}$. The physically meaningful root is approximately

$$Q \approx K_c^2 \{1 - 3\beta\,(1 - K_c^{-1})\} \tag{79}$$

With (62), the total power scaling relation is

$$\frac{P'}{P} \approx K^3 K_c^2 \{1 - 3\beta\,(1 - K_c^{-1})\} \tag{80}$$

The fact that (80) retains some actual physical quantities (ν, N_{Sc}, ϵ, L_c) means that genuine similarity is not possible for this case. Yet, if these quantities are known for the model, this kind of relation can be used to scale.

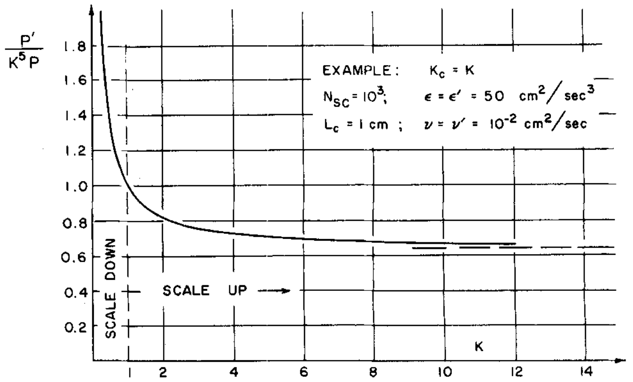

Fig. 7. Particular example of mixer power scaling for $N_{SC} >> 1$.

It is observed that this agrees with (64) in the limit $N_{Sc} = N_{Sc}' = 1$. To illustrate the departure of (80) from this simple limit, consider the following numerical example:

$$K_c = K;\ N_{Sc} = 10^3;\ \epsilon = 50\,\frac{\text{sq. cm.}}{\text{sec.}^3}$$

$$L_c = 1 \text{ cm.};\ \nu = 10^{-2}\,\frac{\text{sq. cm.}}{\text{sec.}}$$

so that

$$P' = K^3 P \left\{ 0.64 + \frac{0.36}{K} \right\}$$

This curve is plotted as Figure 7.

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NOTATION

A = dimensionless, constant coefficient in three-dimensional, inertial subrange, turbulent energy spectrum, Equation (15)
B = dimensionless, constant coefficient in three-dimensional, inertial-convective subrange, contaminant spectrum, Equation (16)
C = concentration of contaminant
c = $\equiv C - \overline{C}$, fluctuation in concentration
D_v = molecular diffusivity of C
$E(k)$ = three-dimensional energy spectrum of turbulence
$f(r)$ = a particular two-point, velocity correlation coefficient function (see p. 46, reference 9). r is the scalar distance between the points
$G(k)$ = three-dimensional spectrum of contaminant fluctuation field
H, H' = characteristic lengths in gross flow systems of mixers
H_c, H_c' = characteristic lengths in contaminant injection systems of mixers
$\underline{\underline{k}}$ = wave number vector (of magnitude k)
k^*, k_K, k_B = maximum characteristic wave number magnitudes for various spectra and conditions, defined in Equations (10), (11), (12), respectively
k_o = wave number characterizing the energy-bearing eddies
${}_ck_o$ = wave number characterizing the principal content of the contaminant fluctuations
K = K^3 is a mass scaling factor, for two mixers, Equation (65), so that for equal densities K is a length scaling factor
K_c = length scaling factor for C fields in the two mixers, Equation (65) or (67)
L = typical integral scale of the turbulent motion roughly the inverse of k_o
L_f = particular integral scale of turbulent motion, defined as the integral of $f(r)$ or by Equation (25) for isotropic turbulence
L_c = corresponding integral scale of the contaminant fluctuations, Equation (26), essentially the inverse of ${}_ck_o$
L_f', L_c' = values of L_f, L_c in a second mixer
l = Taylor type of microscale for the c field, defined by comparing Equations (5) and (7), or in Equation (14)
M, M' = masses of fluid in two different mixers which are geometrically similar, Equation (65)
${}_oN_{Re}$ = turbulence Reynolds number characterizing the energy-bearing eddies, Equation (42)
${}_\lambda N_{Re}$ = turbulence Reynolds number based on the Taylor microscale, defined ahead of Equation (25)
N_{Sc} = Schmidt number, ν/D_v
P, P' = powers used to drive the two mixers
p = static pressure
Q = scaling factor for the turbulent kinetic energy dissipation rates in the two mixers, Equation (71)
R = scaling factor for kinematic viscosities in the two mixers, Equation (71)
t = time
u = magnitude of turbulent velocity vector at a point in space time
u_i = cartesian component of velocity ($i = 1, 2, 3$)
$\underline{\underline{x}}$ = Eulerian space coordinate vector
x_j = cartesian component of coordinate ($j = 1, 2, 3$)

Greek Letters

β = dimensionless constant, defined after Equation (77)
ϵ = turbulent kinetic energy dissipation rate per unit mass
ϵ' = same thing in a second mixer
ϵ_c = rate of destruction of $\overline{c^2}$ by molecular diffusion
η = power efficiency with which a mixing device generates turbulent kinetic energy, Equation (62)
λ_f = Taylor microscale of the turbulence field, defined by comparing Equations (6) and (8), or by Equation (14)
λ = other Taylor microscale, equal to $\lambda_f/\sqrt{2}$ in isotropic turbulence
ν = kinematic viscosity
ν' = kinematic viscosity in another mixer
ρ = density
ρ' = density in another mixer
τ = time interval characterizing the approach to uniformity in a mixing process, Equation (55)

Mathematical Operations

— = mean value
$'$ = root-mean-square $\sqrt{\overline{(\ )^2}}$, except where it refers to parameters in a scaled mixer

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